Theoretical and Mathematical Physics, **152**(1): 933–947 (2007)

BURGERS AND KADOMTSEV–PETVIASHVILI HIERARCHIES: A FUNCTIONAL REPRESENTATION APPROACH

A. Dimakis* **and F. Müller-Hoissen†**

Functional representations of (matrix) Burgers and potential Kadomtsev–Petviashvili (pKP) hierarchies (and others), as well as some corresponding Bäcklund transformations, can be obtained surprisingly simply from a "discrete" functional zero-curvature equation. We use these representations to show that any solution of a Burgers hierarchy is also a solution of the pKP hierarchy. Moreover, the pKP hierarchy can be expressed in the form of an inhomogeneous Burgers hierarchy. In particular, this leads to an extension of the Cole–Hopf transformation to the pKP hierarchy. Furthermore, these hierarchies are solved by the solutions of certain functional Riccati equations.

Keywords: Burgers hierarchy, Cole–Hopf transformation, Kadomtsev–Petviashvili hierarchy, functional Riccati equation

1. Introduction

It was noted in [1] that any solution of the first two equations of the Burgers hierarchy [2]–[10] is also a solution of the potential Kadomtsev–Petviashvili (pKP) equation. The generalization to the case where the dependent variables take their values in a matrix (or, more generally, an associative and typically noncommutative) algebra $\mathbb{A}$ appeared in [11]. It can be easily shown using functional representations (i.e., generating equations depending on auxiliary indeterminates) of the corresponding hierarchies that any solution of the ("noncommutative") Burgers hierarchy indeed also solves the ("noncommutative") pKP hierarchy (see Sec. 4). Moreover, it turns out that the pKP hierarchy can be expressed as an *inhomogeneous Burgers hierarchy*. This means that there is a functional form of the pKP hierarchy involving a matrix function as an inhomogeneous term. If this term is set to zero, then the functional form of the pKP hierarchy reduces to a functional form of the Burgers hierarchy.

Our starting point for generating functional representations of integrable hierarchies is a functional zero-curvature (Zakharov–Shabat) equation, which we recall in Sec. 2 (also see [12], [13]). In Sec. 3, we then treat the simplest nontrivial example: a Burgers hierarchy with the dependent variable in $\mathbb{A}$. We consider another version of the Burgers hierarchy in the appendix. In Sec. 4, we address the case of the pKP hierarchy and its relations to Burgers hierarchies. In particular, we obtain an extension of the Cole–Hopf transformation from the Burgers to the pKP hierarchy, generalizing a result in [11]. In Sec. 5, we show that there is a functional Riccati equation that implies the pKP hierarchy and that reduces under certain conditions to a certain functional Riccati equation that implies the Burgers hierarchy. Because such Riccati equations can be solved explicitly, this offers a quick way to obtain exact solutions. If a "rank-one condition" is imposed (see [14] and the references therein), then these solutions of matrix hierarchies lead to solutions of the corresponding scalar hierarchies.

*Department of Financial and Management Engineering, University of the Aegean, 31 Fostini Str., GR-82100 Chios, Greece.

†Max-Planck-Institut für Dynamik und Selbstorganisation, Bunsenstrasse 10, D-37073 Göttingen, Germany, e-mail: fmuelle@gwdg.de.

Translated from Teoreticheskaya i Matematicheskaya Fizika, Vol. 152, No. 1, pp. 66–82, July, 2007.

2. The functional zero-curvature condition

The integrability conditions of a linear system

$$\partial_{t_n}\psi = B_n\psi, \quad n = 1, 2, \dots, \tag{2.1}$$

with the independent variables $\mathbf{t} := (t_1, t_2, t_3, \dots)$ are the Zakharov–Shabat (zero-curvature) conditions

$$\partial_{t_n}B_m - \partial_{t_m}B_n = [B_n, B_m]. \tag{2.2}$$

We learned [12], [13] that for several important hierarchies, instead of the partial derivatives ∂_{t_n}, it is more convenient to use the operators

$$\hat{\chi}_n := p_n(-\tilde{\partial}), \qquad \tilde{\partial} := \left(\partial_{t_1}, \frac{\partial_{t_2}}{2}, \frac{\partial_{t_3}}{3}, \dots\right), \tag{2.3}$$

where p_n are the elementary Schur polynomials; this insight can be traced back to [15] (also see [16]). In particular, we have

$$\begin{aligned}
&\hat{\chi}_0 = \mathrm{id}, \qquad \hat{\chi}_1 = -\partial_{t_1}, \qquad \hat{\chi}_2 = -\frac{1}{2}\partial_{t_2} + \frac{1}{2}\partial_{t_1}^2,\\
&\hat{\chi}_3 = -\frac{1}{3}\partial_{t_3} + \frac{1}{2}\partial_{t_2}\partial_{t_1} - \frac{1}{6}\partial_{t_1}^3,\\
&\hat{\chi}_4 = -\frac{1}{4}\partial_{t_4} + \frac{1}{3}\partial_{t_3}\partial_{t_1} + \frac{1}{8}\partial_{t_2}^2 - \frac{1}{4}\partial_{t_2}\partial_{t_1}^2 + \frac{1}{24}\partial_{t_1}^4.
\end{aligned}$$

An equivalent form of the above linear system is then

$$\psi_{-[\lambda]} = \mathcal{E}(\lambda)\psi, \tag{2.4}$$

where λ is an indeterminate and $\mathcal{E}(\lambda) = \sum_{n\ge 0}\lambda^n\mathcal{E}_n$ is a formal power series in λ. The coefficients $\mathcal{E}_n$ can be expressed in terms of the B_n, and vice versa. For example, $B_1 = -\mathcal{E}_1$, $B_2 = -2\mathcal{E}_2 - \mathcal{E}_{1,t_1} + \mathcal{E}_1^2$, and $B_3 = -3\mathcal{E}_3 - 3\mathcal{E}_{2,t_1} - \mathcal{E}_{1,t_1t_1} + 2\mathcal{E}_{1,t_1}\mathcal{E}_1 + \mathcal{E}_1\mathcal{E}_{1,t_1} + 3\mathcal{E}_2\mathcal{E}_1 - \mathcal{E}_1^3$. In (2.4), we use the notation $[\lambda] := (\lambda, \lambda^2/2, \lambda^3/3, \dots)$ and

$$f_{-[\lambda]}(\mathbf{t}) := f\big(\mathbf{t} - [\lambda]\big) = \sum_{n=0}^{\infty}\lambda^n\hat{\chi}_n(f) \tag{2.5}$$

(as a formal power series in λ) for any object f dependent on $\mathbf{t}$. This is sometimes called a *Miwa shift*. We also use "positive" Miwa shifts, $f_{[\lambda]}(\mathbf{t}) := f\big(\mathbf{t} + [\lambda]\big) = \sum_{n=0}^{\infty}\lambda^n\chi_n(f)$ with $\chi_n := p_n(\tilde{\partial})$. The integrability conditions are now

$$\mathcal{E}(\lambda)_{-[\mu]}\mathcal{E}(\mu) = \mathcal{E}(\mu)_{-[\lambda]}\mathcal{E}(\lambda), \tag{2.6}$$

where λ and μ are indeterminates. If $\mathcal{E}(\lambda)$ is regarded as a parallel transport operator, then (2.6) can be interpreted as a "discrete" zero-curvature condition, as depicted in the (commutative) diagram

$$\begin{array}{ccc}
\bullet & \xrightarrow{\mathcal{E}(\lambda)} & \bullet \\
{\scriptstyle \mathcal{E}(\mu)}\big\downarrow & & \big\downarrow{\scriptstyle \mathcal{E}(\mu)_{-[\lambda]}} \\
\bullet & \xrightarrow[\mathcal{E}(\lambda)_{-[\mu]}]{} & \bullet
\end{array}.$$

Here, "discrete" is used in the sense of [16] (also see [17], [18] for an approach to integrable equations via discrete zero-curvature equations).

Introducing a "discrete" gauge potential (see [19], [20]) by

$$\mathcal{E}(\lambda) = I - \lambda\mathcal{A}(\lambda), \tag{2.7}$$

where I is the unit element of the (typically matrix) algebra from which the coefficients of the formal power series $\mathcal{A}(\lambda)$ are taken,[1] we can write (2.6) as

$$\Upsilon(\lambda,\mu) = \Upsilon(\mu,\lambda), \qquad \Upsilon(\lambda,\mu) := \mu^{-1}\big(\mathcal{A}(\lambda) - \mathcal{A}(\lambda)_{-[\mu]}\big) + \mathcal{A}(\lambda)_{-[\mu]}\mathcal{A}(\mu). \tag{2.8}$$

Equation (2.6) has the gauge invariance[2]

$$\mathcal{E}(\lambda) \quad\mapsto\quad \mathcal{B}_{-[\lambda]}\mathcal{E}(\lambda)\mathcal{B}^{-1} = \mathcal{E}'(\lambda) \tag{2.9}$$

with an invertible $\mathcal{B}$. This originates from the transformation $\psi' = \mathcal{B}\psi$ of linear system (2.4). In particular, Bäcklund (or Darboux) transformations thus arise (see, e.g., [21]). In terms of the gauge potential, (2.9) is

$$\lambda^{-1}\big(\mathcal{B} - \mathcal{B}_{-[\lambda]}\big) = \mathcal{A}'(\lambda)\mathcal{B} - \mathcal{B}_{-[\lambda]}\mathcal{A}(\lambda). \tag{2.10}$$

In Sec. 3, a Burgers hierarchy results from the simplest nontrivial ansatz for $\mathcal{E}(\lambda)$ (also see the appendix for another version of the matrix Burgers hierarchy). If the gauge potential is linear in the operator of partial differentiation with respect to the variable x, we obtain the pKP hierarchy (see Sec. 4). There are more examples (also see [12], [13]) and a generalization of (2.6) that covers multicomponent hierarchies.

3. The Burgers hierarchy and Cole–Hopf and Bäcklund transformations in functional form

We choose

$$\mathcal{E}(\lambda) = I - \lambda\phi, \tag{3.1}$$

and the gauge potential $\mathcal{A}(\lambda) = \phi$ is hence independent of λ. We can then express (2.6) as

$$\omega(\lambda) = \omega(\mu), \qquad \omega(\lambda) := \lambda^{-1}\big(\phi - \phi_{-[\lambda]}\big) + \phi_{-[\lambda]}\phi. \tag{3.2}$$

Because $\lim_{\lambda\to 0}\omega(\lambda) = \phi_x + \phi^2$, where $x := t_1$, this turns out to be equivalent to

$$\Omega(\lambda) := \omega(\lambda) - \phi_x - \phi^2 \equiv \big(\phi - \phi_{-[\lambda]}\big)(\lambda^{-1} - \phi) - \phi_x = 0, \tag{3.3}$$

which is a functional representation of a ("noncommutative") Burgers hierarchy. The first hierarchy equation is the Burgers equation $\phi_y = \phi_{xx} + 2\phi_x\phi$, where $y := t_2$. From (3.1), we obtain $B_1 = \phi$, $B_2 = \phi_{t_1} + \phi^2$, $B_3 = \phi_{t_1t_1} + 2\phi_{t_1}\phi + \phi\phi_{t_1} + \phi^3$, and so on. Zakharov–Shabat equations (2.2) then also produce the Burgers hierarchy equations.

Because the curvature vanishes, we can expect that there is a gauge in which the gauge potential $\mathcal{A}$ vanishes. Hence, we seek an invertible f such that

$$f^{-1}_{-[\lambda]}\mathcal{E}(\lambda)f = I \tag{3.4}$$

(i.e., $\mathcal{E}'(\lambda) = I$ and $\mathcal{B} = f^{-1}$ in (2.9)), which is

$$\lambda^{-1}\big(f - f_{-[\lambda]}\big) = \phi f. \tag{3.5}$$

[1] More generally, the coefficients of the formal power series $\mathcal{E}(\lambda)$ and $\mathcal{A}(\lambda)$ can be elements of any unital associative algebra $\mathbb{A}$ whose elements are differentiable with respect to the set of coordinates $\mathbf{t}$ (which requires a Banach space structure on $\mathbb{A}$). Then ψ is an element of a left $\mathbb{A}$-module.

[2] Transformation (2.9) extends the above planar diagram to a "commutative cube," where $\mathcal{B}$ acts along the orthogonal bonds.

Proposition 1. *Equation* (3.5) *is a functional representation of the Cole–Hopf transformation*

$$\phi = f_x f^{-1}, \tag{3.6}$$

$$\partial_{t_n} f = \partial_x^n f, \quad n = 2, 3, \dots. \tag{3.7}$$

Any invertible f that solves linear "heat hierarchy" (3.7) *determines a solution of the Burgers hierarchy via* (3.6).[3]

This noncommutative version of the Cole–Hopf transformation (see, e.g., [3], [7], [11], [22]–[26]) for the Burgers equation appeared, for instance, in [3], [26]–[28].

Proof. A well-known identity for the elementary Schur polynomials p_n leads to

$$n\hat{\chi}_n = -\sum_{k=1}^{n} \partial_{t_k} \hat{\chi}_{n-k} = -\sum_{k=1}^{n-2} \partial_{t_k} \hat{\chi}_{n-k} - \partial_{t_n} + \partial_x \partial_{t_{n-1}}, \quad n = 2, 3, \dots.$$

Using this, we prove by induction that for an arbitrary integer $N > 1$, the first N equations of the system in (3.7) are equivalent to the first N equations $\hat{\chi}_n(f) = 0$, $n = 2, 3, \dots$. Together with (3.6), these equations are equivalent to (3.5). Furthermore, the integrability condition for (3.5) is Burgers hierarchy equation (3.2).

Remark 1. Special solutions of heat hierarchy (3.7) are given by arbitrary linear combinations of the Schur polynomials $p_n(\mathbf{t})$, $n = 0, 1, 2, \dots$, with constant coefficients in $\mathbb{A}$. In particular, with constant $P \in \mathbb{A}$,

$$e^{\xi(P)} = \sum_{n \geq 0} p_n(\mathbf{t}) P^n, \qquad \xi(P) := \sum_{m \geq 1} t_m P^m \tag{3.8}$$

is a (formal) solution.

Transformation equation (2.10) is now

$$\lambda^{-1}\bigl(\mathcal{B} - \mathcal{B}_{-[\lambda]}\bigr) = \phi' \mathcal{B} - \mathcal{B}_{-[\lambda]} \phi. \tag{3.9}$$

In the limit as $\lambda \to 0$, this implies

$$\phi' = \mathcal{B} \phi \mathcal{B}^{-1} + \mathcal{B}_x \mathcal{B}^{-1}. \tag{3.10}$$

Using this equation to eliminate ϕ' from (3.9) yields

$$\bigl(\mathcal{B} - \mathcal{B}_{-[\lambda]}\bigr)(\lambda^{-1} - \phi) = \mathcal{B}_x. \tag{3.11}$$

Together with (3.10), this is equivalent to (3.9). Any invertible $\mathcal{B}$ that satisfies (3.11) generates a new solution ϕ' from a given solution ϕ of the Burgers hierarchy via (3.10). Because (3.11) is linear in $\mathcal{B}$, linear combinations of solutions (with constant left coefficients) are again solutions of (3.11). Comparing (3.11) with (3.3) shows that $\mathcal{B} = \phi$ is a particular solution. Obviously, any constant element α also satisfies (3.11). Hence, $\mathcal{B} = \alpha + \beta\phi$ with arbitrary constants α and β satisfies these conditions, and (3.10) becomes

$$\phi' = (\alpha + \beta\phi)\phi(\alpha + \beta\phi)^{-1} + \beta\phi_x(\alpha + \beta\phi)^{-1} \tag{3.12}$$

assuming that the inverse exists. This covers the elementary Bäcklund (or Darboux) transformations obtained in [3], [9], [24], [29], [30].

---

[3] Conversely, if ϕ solves the Burgers hierarchy, then we choose f such that $f_x = \phi f$. Then

$$0 = \Omega(\lambda) f = \bigl(\partial_x - \phi_{-[\lambda]}\bigr)\bigl[\lambda^{-1}\bigl(f - f_{-[\lambda]}\bigr) - f_x\bigr]$$

implies that f solves the heat hierarchy if $\partial_x - \phi_{-[\lambda]}$ is invertible.

4. The potential KP hierarchy in functional form and relations to the Burgers hierarchy

If we choose[4]

$$\mathcal{E}(\lambda) = I - \lambda\big(w(\lambda) + \partial\big), \tag{4.1}$$

i.e., $\mathcal{A}(\lambda) = w(\lambda) + \partial$, where $\partial = \partial_x$, then (2.8) leads to the equations

$$\begin{aligned}\lambda^{-1}\big(w(\mu) - w(\mu)_{-[\lambda]}\big) + w(\mu)_{-[\lambda]} w(\lambda) + w(\lambda)_x = \\ = \mu^{-1}\big(w(\lambda) - w(\lambda)_{-[\mu]}\big) + w(\lambda)_{-[\mu]} w(\mu) + w(\mu)_x\end{aligned} \tag{4.2}$$

and

$$w(\lambda) - w(\lambda)_{-[\mu]} = w(\mu) - w(\mu)_{-[\lambda]}. \tag{4.3}$$

Equation (4.3) is solved by

$$w(\lambda) = \phi - \phi_{-[\lambda]}, \tag{4.4}$$

and Eq. (4.2) then becomes

$$\omega(\lambda)_{-[\mu]} - \omega(\mu)_{-[\lambda]} = \omega(\lambda) - \omega(\mu) - (\phi_x + \phi^2)_{-[\lambda]} + (\phi_x + \phi^2)_{-[\mu]} \tag{4.5}$$

with the definition in (3.2) taken into account. Summing this expression three times with cyclically permuted indeterminates λ_1, λ_2, and λ_3 results in the Bogdanov–Konopelchenko (BK) functional equation [31], [32],

$$\sum_{i,j,k=1}^{3} \epsilon_{ijk}\, \omega(\lambda_i)_{-[\lambda_j]} = 0, \tag{4.6}$$

where ϵ_{ijk} is totally antisymmetric with $\epsilon_{123} = 1$. This determines the pKP hierarchy and is equivalent to (4.5). Expanding (4.5) in λ and μ yields $\partial_x \phi = \partial_{t_1}\phi$ and

$$\hat{\chi}_m \hat{\chi}_{n+1}(\phi) - \hat{\chi}_n \hat{\chi}_{m+1}(\phi) = \hat{\chi}_m\big(\hat{\chi}_n(\phi)\phi\big) - \hat{\chi}_n\big(\hat{\chi}_m(\phi)\phi\big), \quad m, n = 1, 2, \ldots. \tag{4.7}$$

An equivalent expression for the pKP hierarchy (in the scalar case) already appeared in [33] (also see [10], [12]). For $m = 1$ and $n = 2$, this yields the pKP equation

$$(4\phi_t - \phi_{xxx} - 6{\phi_x}^2)_x - 3\phi_{yy} + 6[\phi_x, \phi_y] = 0, \tag{4.8}$$

where $x = t_1$, $y = t_2$, and $t = t_3$. Comparing (3.2) with (4.5) shows that any solution of the Burgers hierarchy considered in Sec. 3 also solves the pKP hierarchy.

Remark 2. There is a (Sato–Wilson) pseudodifferential operator $W = I + \sum_{n>0} w_n \partial^{-n}$ such that $\mathcal{B} = W^{-1}$ in (2.9) transforms $\mathcal{E}(\lambda)$ into $\mathcal{E}'(\lambda) = I - \lambda\partial$. It is determined (up to multiplication by a constant operator $I + \sum_{n>0} c_n \partial^{-n}$) by

$$w_1 - w_{1,-[\lambda]} = \phi_{-[\lambda]} - \phi, \qquad w_{n+1} - w_{n+1,-[\lambda]} = \lambda^{-1}\big(w_n - w_{n,-[\lambda]}\big) - w_{n,x} - \big(\phi - \phi_{-[\lambda]}\big) w_n.$$

[4]Starting instead with $\mathcal{E}(\lambda) = I - \lambda v(\lambda)\partial$ leads in the same way to the modified KP hierarchy [12]. The two choices of $\mathcal{E}(\lambda)$ are related by a gauge transformation (Miura transformation).

4.1. The pKP hierarchy as an inhomogeneous Burgers hierarchy. We observe that (4.5) can also be written as

$$\Omega(\mu) - \Omega(\mu)_{-[\lambda]} = \Omega(\lambda) - \Omega(\lambda)_{-[\mu]}, \tag{4.9}$$

where $\Omega(\lambda)$ is the expression defined in (3.3) in terms of ϕ. As a consequence, the pKP hierarchy becomes

$$\Omega(\lambda) = \theta - \theta_{-[\lambda]} \tag{4.10}$$

with some θ. If the right-hand side vanishes, i.e., if θ is constant, then this is precisely functional representation (3.3) of the Burgers hierarchy considered in Sec. 3. Representation (4.10) is equivalent to

$$\hat{\chi}_{n+1}(\phi) - \hat{\chi}_n(\phi)\phi = \hat{\chi}_n(\theta), \quad n = 1, 2, \ldots. \tag{4.11}$$

The first two equations are

$$\begin{aligned} \phi_y &= \phi_{xx} + 2\phi_x\phi + 2\theta_x, \\ \phi_t &= \phi_{xxx} + 3\phi_{xx}\phi + 3{\phi_x}^2 + 3\phi_x\phi^2 + 3\theta_x\phi + \frac{3}{2}(\theta_y + \theta_{xx}) \end{aligned} \tag{4.12}$$

after we use the first equation to replace ϕ_y in the second. For constant θ, these are the first two equations of the Burgers hierarchy. Eliminating θ from (4.12), we recover pKP equation (4.8).

Applying a Miwa shift to (4.5) leads to

$$\widetilde{\omega}(\lambda)_{[\mu]} - \widetilde{\omega}(\mu)_{[\lambda]} = \widetilde{\omega}(\lambda) - \widetilde{\omega}(\mu) - (\phi_x + \phi^2)_{[\lambda]} + (\phi_x + \phi^2)_{[\mu]}, \tag{4.13}$$

where

$$\widetilde{\omega}(\lambda) := \omega(\lambda)_{[\lambda]} = \lambda^{-1}\big(\phi_{[\lambda]} - \phi\big) + \phi\phi_{[\lambda]}.$$

Because this can be written as

$$\widetilde{\Omega}(\lambda)_{[\mu]} - \widetilde{\Omega}(\lambda) = \widetilde{\Omega}(\mu)_{[\lambda]} - \widetilde{\Omega}(\mu) \tag{4.14}$$

with

$$\widetilde{\Omega}(\lambda) := \widetilde{\omega}(\lambda) - \phi_x - \phi^2 = (\lambda^{-1} + \phi)\big(\phi_{[\lambda]} - \phi\big) - \phi_x,$$

the pKP hierarchy can also be expressed as

$$\widetilde{\Omega}(\lambda) = \tilde{\theta}_{[\lambda]} - \tilde{\theta} \tag{4.15}$$

with some $\tilde{\theta}$. The functions $\tilde{\theta}$ and θ are related by $\tilde{\theta} - \theta = \phi_x + \phi^2$. If $\tilde{\theta}$ is constant (and the right-hand side hence vanishes), then the last equation reduces to the "opposite" Burgers hierarchy (see the appendix)

$$(\lambda^{-1} + \phi)\big(\phi_{[\lambda]} - \phi\big) = \phi_x, \tag{4.16}$$

which starts with $\phi_y = -\phi_{xx} - 2\phi\phi_x$. In particular, we have the following result.

Proposition 2. *Any solution of either of the two Burgers hierarchies also solves the pKP hierarchy.*

4.2. A Cole–Hopf transformation for the matrix pKP hierarchy.

Theorem. *Let* $(\mathbb{A},\,\cdot\,)$ *be the algebra of* $M{\times}N$ *matrices of functions of* $\mathbf{t}$ *with the product*

$$A\cdot B = AQB, \tag{4.17}$$

where the ordinary matrix product is used in the right-hand side and Q *is a constant* $N{\times}M$ *matrix. Let* X *be an invertible* $N{\times}N$ *matrix and* $Y\in\mathbb{A}$ *be such that* X *and* Y *solve linear heat hierarchy* (3.7) *and satisfy*

$$X_x = RX + QY \tag{4.18}$$

with a constant $N{\times}N$ *matrix* R*. Then the pKP hierarchy in* $(\mathbb{A},\,\cdot\,)$ *is solved by*

$$\phi := YX^{-1}. \tag{4.19}$$

Proof. Using (4.19), we can write the expression $\Omega(\lambda)$ defined in (3.3) (where a factor Q enters the nonlinear term because of (4.17)) as

$$\Omega(\lambda) = \big(\phi-\phi_{-[\lambda]}\big)(X_x-QY)X^{-1} + \big(\lambda^{-1}\big(Y-Y_{-[\lambda]}\big)-Y_x\big)X^{-1} - \\ - \phi_{-[\lambda]}\big(\lambda^{-1}\big(X-X_{-[\lambda]}\big)-X_x\big)X^{-1}.$$

If X and Y solve the heat hierarchy, then $\hat{\chi}_n(X)=0=\hat{\chi}_n(Y)$, $n=2,3,\dots$, and hence

$$\lambda^{-1}\big(X-X_{-[\lambda]}\big)=X_x, \qquad \lambda^{-1}\big(Y-Y_{-[\lambda]}\big)=Y_x.$$

Using these equations, we reduce the above expression for $\Omega(\lambda)$ to

$$\Omega(\lambda) = \big(\phi-\phi_{-[\lambda]}\big)(X_x-QY)X^{-1}.$$

If $R:=(X_x-QY)X^{-1}$ is constant, which means that (4.18) holds, then this takes form (4.10) of the pKP hierarchy with $\theta=\phi R$.[5] Therefore, ϕ solves the pKP hierarchy.

If $R=0$, then (4.18) and (4.19) with $M=N$ and $Q=I_N$ reduce to $\phi=X_xX^{-1}$, and we recover the Cole–Hopf transformation for the Burgers hierarchy. We note that the conditions imposed on X already imply $Q\big(\lambda^{-1}\big(Y-Y_{-[\lambda]}\big)-Y_x\big)=0$ and Y therefore automatically satisfies the heat hierarchy if Q has the maximum rank. Furthermore, if we consider $Q\phi$ instead of ϕ, then the assumption on Y is unnecessary.

Corollary. *Let* X *solve the heat hierarchy and* (4.18) *with some* Y*. Then* $Q\phi$ *with* ϕ *given by* (4.19) *solves the* $(N{\times}N)$*-matrix pKP hierarchy with the usual matrix product.*

A similar result appeared already in [11] for the case where $\operatorname{rank}Q=1$ (see [14] and the references therein). Then $\operatorname{tr}(QA\cdot B)=\operatorname{tr}(QA)\operatorname{tr}(QB)$; hence, by (4.18), the function

$$\varphi := \operatorname{tr}(Q\phi) = -\operatorname{tr}R + (\log\tau)_x, \qquad \tau := \det X, \tag{4.20}$$

solves the *scalar* pKP hierarchy.

[5] We also note that $\tilde{\theta}=\theta+\phi_x+\phi Q\phi=Y_xX^{-1}$ by (4.18).

4.3. Bäcklund and Darboux transformations. Substituting the ansatz $\mathcal{B} = b(\mathbf{t}) - \partial$ in (2.10) leads to the two equations

$$b - \phi' + \phi = (b - \phi' + \phi)_{-[\lambda]} \tag{4.21}$$

and

$$\lambda^{-1}\big(b - b_{-[\lambda]}\big) - b_x = \big(\phi' - \phi'_{-[\lambda]}\big)b - b_{-[\lambda]}\big(\phi - \phi_{-[\lambda]}\big) + \big(\phi - \phi_{-[\lambda]}\big)_x. \tag{4.22}$$

The solution of (4.21) is

$$b = \phi' - \phi \tag{4.23}$$

(an additive constant is absorbed into ϕ'). Equation (4.22) can then be written as

$$\Omega(\lambda) - \Omega'(\lambda) = \Gamma(\phi, \phi') - \Gamma(\phi, \phi')_{-[\lambda]}, \tag{4.24}$$

where $\Omega'(\lambda)$ is built with ϕ' and $\Gamma(\phi, \phi') := (\phi' - \phi)\phi - \phi_x$. This is an elementary Bäcklund transformation (BT) of the pKP hierarchy. Extending the above ansatz for $\mathcal{B}$ to the nth order in ∂ leads to equations determining nth-order BTs. These are solved by an n-fold product of elementary BTs.

Using (4.10), we find

$$0 = \Gamma(\phi, \phi') + \theta' - \theta = \phi'\phi + \theta' - \tilde{\theta}. \tag{4.25}$$

Let $\mathcal{B}_{n,m}$ denote the BT taking a pKP solution ϕ_m to a new solution ϕ_n. The permutability relation[6] $\mathcal{B}_{(3,1)}\mathcal{B}_{(1,0)} = \mathcal{B}_{(3,2)}\mathcal{B}_{(2,0)}$ then results in

$$(\phi_2 - \phi_1)_x = \phi_3(\phi_2 - \phi_1) + (\phi_2 - \phi_1)\phi_0 + \phi_1^2 - \phi_2^2. \tag{4.26}$$

In the commutative scalar case, setting $\phi = \tau_x/\tau$ with a function τ yields $\tau_0\tau_3 = \tau_1\tau_{2,x} - \tau_{1,x}\tau_2$. Relation (4.26) algebraically determines a new solution ϕ_3 in terms of a given solution ϕ_0 and the corresponding Bäcklund descendants ϕ_1 and ϕ_2.

In the case under consideration, linear system (2.4) becomes

$$\lambda^{-1}\big(\psi - \psi_{-[\lambda]}\big) - \psi_x = \big(\phi - \phi_{-[\lambda]}\big)\psi \tag{4.27}$$

(see [15] for an equivalent version in the scalar case). If ψ is invertible, then we obtain

$$\phi - \phi_{-[\lambda]} = \lambda^{-1}\big(\psi - \psi_{-[\lambda]}\big)\psi^{-1} - \psi_x\psi^{-1}. \tag{4.28}$$

Using (4.23) to eliminate ϕ' from (4.22) and then using (4.28) to eliminate $\phi - \phi_{-[\lambda]}$ results in

$$(b - \psi_x\psi^{-1})_x + (b - \psi_x\psi^{-1})\big(b + \lambda^{-1}\psi_{-[\lambda]}\psi^{-1}\big) - \\ - \big(b_{-[\lambda]} + \lambda^{-1}\psi_{-[\lambda]}\psi^{-1}\big)(b - \psi_x\psi^{-1}) = 0. \tag{4.29}$$

This equation is obviously solved by

$$b = \psi_x\psi^{-1}. \tag{4.30}$$

Hence, if ψ_1 solves the linear system with a solution ϕ of the pKP hierarchy, then

$$\phi' = \phi + \psi_{1,x}\psi_1^{-1} \tag{4.31}$$

is a new solution of the pKP hierarchy.[7] This is a Darboux transformation [30], [34]–[36].

[6] We note that this is also a discrete zero-curvature condition.

[7] Moreover, $\psi' = \mathcal{B}\psi = \psi_x - \psi_{1,x}\psi_1^{-1}\psi$ satisfies the linear system with ϕ'.

5. Functional Riccati equations associated with KP and Burgers hierarchies: Toward exact solutions

We consider BK functional equation (4.6) in the algebra $(\mathbb{A}, \cdot)$, where $\mathbb{A}$ is the set of $M\times N$ matrices of complex functions of $\mathbf{t}$, supplied with product (4.17). The simplest nontrivial equation, which results from this formula by expanding in powers of the indeterminates, is the matrix pKP equation

$$(4\phi_t - \phi_{xxx} - 6\phi_x Q\phi_x)_x = 3\phi_{yy} - 6(\phi_x Q\phi_y - \phi_y Q\phi_x). \tag{5.1}$$

As a consequence, ϕQ satisfies the $(M\times M)$-matrix pKP hierarchy, and $Q\phi$ satisfies the $(N\times N)$-matrix pKP hierarchy. Moreover, if $Q = VU^{\mathrm{T}}$ with an $N\times m$ matrix V and an $M\times m$ matrix U, then $U^{\mathrm{T}}\phi V$ satisfies the $(m\times m)$-matrix pKP hierarchy. In particular, for $m = 1$, this becomes the scalar pKP hierarchy, and Q has the rank one.

The crucial observation now is that the BK functional equation and hence the pKP hierarchy are satisfied if ϕ solves

$$\omega(\lambda) = S + L\phi - \phi_{-[\lambda]}R \tag{5.2}$$

with constant matrices S, L, and R with the respective dimensions $M\times N$, $M\times M$, and $N\times N$. This is a functional matrix Riccati equation for ϕ,

$$\lambda^{-1}\bigl(\phi - \phi_{-[\lambda]}\bigr) = S + L\phi - \phi_{-[\lambda]}R - \phi_{-[\lambda]}Q\phi. \tag{5.3}$$

The integrability condition for this functional equation is satisfied[8] because

$$\begin{aligned}\bigl(\phi_{-[\lambda]}\bigr)_{-[\mu]} &= \bigl[(\lambda^{-1} - L)\phi_{-[\mu]} - S\bigr]\bigl[(\lambda^{-1} - R) - Q\phi_{-[\mu]}\bigr]^{-1} = \\ &= \bigl[(\lambda^{-1} - L)(\mu^{-1} - L)\phi - (\lambda^{-1} - \mu^{-1})S + LS + SR + SQ\phi\bigr] \times \\ &\quad \times \bigl[(\lambda^{-1} - R)(\mu^{-1} - R) - (\lambda^{-1} + \mu^{-1})Q\phi + (RQ + QL)\phi + QS\bigr]^{-1}\end{aligned} \tag{5.4}$$

is symmetric in λ and μ and therefore equals $\bigl(\phi_{-[\mu]}\bigr)_{-[\lambda]}$. The Riccati equation implies

$$\Omega(\lambda) = \bigl(\phi - \phi_{-[\lambda]}\bigr)R, \qquad \widetilde{\Omega}(\lambda) = L\bigl(\phi_{[\lambda]} - \phi\bigr). \tag{5.5}$$

This shows that with $R = 0$ or $L = 0$, any solution of (5.3) also solves the respective Burgers hierarchy (3.3) or opposite Burgers hierarchy (4.16) in $(\mathbb{A}, \cdot)$.

It is well known that matrix Riccati equations can be linearized [37], [38]. This is achieved by regarding $\phi(\mathbf{t})$ as an element of the Grassmannian $G(N, N+M)$ of N-dimensional linear subspaces of $\mathbb{C}^{N+M}$ via $\kappa(\phi) = \operatorname{span}(I_N, \phi^{\mathrm{T}})^{\mathrm{T}}$ because $\kappa^{-1}\colon G(N, N+M) \to \mathbb{C}^{M\times N}$ defines a chart for the manifold $G(N, N+M)$. In fact, (5.3) can be lifted to a linear equation on the space of $(N+M)\times N$ matrices:

$$\lambda^{-1}\bigl(Z - Z_{-[\lambda]}\bigr) = HZ, \tag{5.6}$$

where

$$Z = \begin{pmatrix} X \\ Y \end{pmatrix}, \qquad H = \begin{pmatrix} R & Q \\ S & L \end{pmatrix}. \tag{5.7}$$

Hence,

$$\lambda^{-1}(X - X_{-[\lambda]}) = RX + QY, \qquad \lambda^{-1}(Y - Y_{-[\lambda]}) = SX + LY. \tag{5.8}$$

[8] This also follows from our work in [10] and is the reason for the choice of the right-hand side of (5.2).

If X is invertible and

$$\phi = Y X^{-1}, \tag{5.9}$$

then these equations imply

$$\phi_{-[\lambda]} = Y_{-[\lambda]} X^{-1}_{-[\lambda]} = \big[\phi - \lambda(S + L\phi)\big] \big[I_N - \lambda(R + Q\phi)\big]^{-1}, \tag{5.10}$$

which is (5.3). Therefore, any solution Z of linear functional equation (5.6) with an invertible X determines a solution (5.9) of functional matrix Riccati equation (5.3) and hence a solution of the matrix pKP hierarchy we started with.

Remark 3. The first equation in (5.8) is equivalent to (4.18) and the heat hierarchy for X. Because the second equation in (5.8) implies that Y must also solve the heat hierarchy, the ϕ determined by (5.9) already solves the pKP hierarchy by the theorem in Sec. 4.2 without using the additional equation $Y_x = SX + LY$, which results from the second equation in (5.8), but this equation helps select interesting classes of solutions (see below). In any case, the Riccati approach corresponds to a class of (generalized) Cole–Hopf transformations in the sense of the theorem above. We also note that $\tilde{\theta} = S + L\phi$.

The general solution of (5.6) is

$$Z = e^{\xi(H)} Z_0, \qquad \xi(H) = \sum_{n\geq 1} H^n t_n, \qquad Z_0 = \begin{pmatrix} X_0 \\ Y_0 \end{pmatrix}, \tag{5.11}$$

where X_0 is invertible. As a consequence, $Z_{t_n} = H^n Z$. Setting

$$e^{\xi(H)} =: \begin{pmatrix} \Xi_{11} & \Xi_{12} \\ \Xi_{21} & \Xi_{22} \end{pmatrix}, \tag{5.12}$$

we have

$$\phi = (\Xi_{21} + \Xi_{22}\phi_0)(\Xi_{11} + \Xi_{12}\phi_0)^{-1}, \tag{5.13}$$

where $\phi_0 = Y_0 X_0^{-1}$. This is a matrix fractional transformation with coefficients depending on $\mathbf{t}$. For any choice of the matrices S, L, R, and Q, this ϕ is a solution of the pKP hierarchy in the matrix algebra with product (4.17). The practical problem is to compute $e^{\xi(H)}$ explicitly.

Remark 4. With $Z = e^{\xi(H)} Z_0$, TZ also satisfies (5.6) if T is constant and commutes with H. In particular, $T = kI_{M+N} + H$ with any constant k induces such a transformation. It results in the matrix fractional transformation (with constant coefficients) $\phi' = (S + L'\phi)(R' + Q\phi)^{-1}$, where $L' := L + kI_M$ and $R' := R + kI_N$.

Example 1. Let $S = 0$ and $Q = RK - KL$ with a constant $N\times M$ matrix K. Then we have

$$H^n = \begin{pmatrix} R^n & R^n K - K L^n \\ 0 & L^n \end{pmatrix}, \qquad \xi(H) = \begin{pmatrix} \xi(R) & \xi(R)K - K\xi(L) \\ 0 & \xi(L) \end{pmatrix}, \tag{5.14}$$

and hence

$$e^{\xi(H)} = \begin{pmatrix} e^{\xi(R)} & e^{\xi(R)}K - Ke^{\xi(L)} \\ 0 & e^{\xi(L)} \end{pmatrix}. \tag{5.15}$$

Therefore, (5.13) becomes

$$\phi = e^{\xi(L)}\phi_0(I_N + K\phi_0 - e^{-\xi(R)}Ke^{\xi(L)}\phi_0)^{-1}e^{-\xi(R)}. \tag{5.16}$$

If Q has rank one, then we obtain the scalar pKP hierarchy solution

$$\begin{aligned}\varphi &= \operatorname{tr}(Q\phi) = \operatorname{tr}\log\bigl(I_N + K\phi_0 - e^{-\xi(R)}Ke^{\xi(L)}\phi_0\bigr)_x = (\log\tau)_x,\\ \tau &= \det\bigl(I_N + K\phi_0 - e^{-\xi(R)}Ke^{\xi(L)}\phi_0\bigr),\end{aligned} \tag{5.17}$$

which includes well-known formulas for KP multisolitons [39] and resonances (see, e.g., [40], [41] and the references therein).

Example 2. Let $M = N$, and let $L = S\pi_-$, $R = \pi_+S$, and $Q = \pi_+S\pi_-$ with constant $N\times N$ matrices S and $\pi_\pm$ such that $\pi_+ + \pi_- = I_N$. It is easy to see that

$$H^n = \begin{pmatrix}\pi_+S^n & \pi_+S^n\pi_-\\ S^n & S^n\pi_-\end{pmatrix}. \tag{5.18}$$

As a consequence, we obtain

$$e^{\xi(H)} = \begin{pmatrix}\pi_- + \pi_+e^{\xi(S)} & \pi_+(e^{\xi(S)} - I_N)\pi_-\\ e^{\xi(S)} - I_N & \pi_+ + e^{\xi(S)}\pi_-\end{pmatrix}, \tag{5.19}$$

and (5.13) is

$$\phi = (-A + e^{\xi(S)}B)(\pi_-A + \pi_+e^{\xi(S)}B)^{-1}, \tag{5.20}$$

where $A := I_N - \pi_+\phi_0$ and $B := I_N + \pi_-\phi_0$. If $\operatorname{rank}(\pi_+S\pi_-) = 1$, then

$$\varphi = \operatorname{tr}(Q\phi) = -\operatorname{tr}(\pi_+S) + (\log\tau)_x, \qquad \tau = \det\bigl(\pi_-A + \pi_+e^{\xi(S)}B\bigr). \tag{5.21}$$

For example, we let $N = m + n$ and choose

$$\pi_- = \begin{pmatrix}I_m & 0\\ 0 & 0\end{pmatrix}, \qquad \pi_+ = \begin{pmatrix}0 & 0\\ 0 & I_n\end{pmatrix}. \tag{5.22}$$

We set

$$\phi_0 = \begin{pmatrix}(\phi_0)_{--} & (\phi_0)_{-+}\\ (\phi_0)_{+-} & (\phi_0)_{++}\end{pmatrix}, \qquad S = \begin{pmatrix}S_{--} & S_{-+}\\ S_{+-} & S_{++}\end{pmatrix}. \tag{5.23}$$

Because $\operatorname{rank} Q = 1$ means $\operatorname{rank} S_{+-} = 1$ (also see [14]), we obtain

$$\tau = \det\bigl((e^{\xi(S)})_{++} + (e^{\xi(S)})_{+-}(\phi_0)_{-+}\bigr). \tag{5.24}$$

In particular, if S is the shift operator $Se_i = e_{i+1}$, then this determines τ-functions, which can be expressed in terms of Schur polynomials. This corresponds to a finite version of the Sato theory (see [14]). For example, if $m = n = 2$ and $(\phi_0)_{-+} = \left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right)$, then we obtain

$$\tau = 1 + cx + a\left(y + \frac{x^2}{2}\right) + d\left(y - \frac{x^2}{2}\right) + b\left(t - \frac{x^3}{3}\right) + (ad - bc)\left(-xt + y^2 + \frac{x^4}{12}\right).$$

Appendix: Opposite Burgers hierarchy and beyond

We generalize the ansatz for $\mathcal{E}(\lambda)$ considered in Sec. 3 to

$$\mathcal{E}(\lambda) = I - \lambda \sum_{n\geq 0} \lambda^n \phi_n. \tag{A.1}$$

Then (2.8) becomes

$$\hat{\chi}_{n+1}(\phi_m) - \hat{\chi}_{m+1}(\phi_n) = \sum_{k=0}^{n} \hat{\chi}_k(\phi_m)\phi_{n-k} - \sum_{k=0}^{m} \hat{\chi}_k(\phi_n)\phi_{m-k}, \tag{A.2}$$

where $m, n = 0, 1, 2, \ldots$. This is an infinite system of coupled equations. As in Sec. 3, we seek a gauge transformation such that condition (3.4) is satisfied, which is

$$\lambda^{-1}\big(f - f_{-[\lambda]}\big) = \sum_{n\geq 0} \lambda^n \phi_n f. \tag{A.3}$$

Expanding the left-hand side in powers of λ, we obtain a generalized Cole–Hopf transformation,

$$\phi_0 = f_x f^{-1}, \qquad \phi_n = -\hat{\chi}_{n+1}(f) f^{-1}, \quad n = 1, 2, \ldots. \tag{A.4}$$

By construction, this solves the zero-curvature equation and hence hierarchy (A.2). Gauge transformation (2.10) becomes

$$\lambda^{-1}\big(\mathcal{B} - \mathcal{B}_{-[\lambda]}\big) = \sum_{n=0}^{\infty} \lambda^n (\phi'_n \mathcal{B} - \mathcal{B}_{-[\lambda]}\phi_n), \tag{A.5}$$

and hence

$$\begin{aligned} \phi'_0 &= \mathcal{B}\phi_0\mathcal{B}^{-1} + \mathcal{B}_x\mathcal{B}^{-1}, \\ \hat{\chi}_{n+1}(\mathcal{B}) &= -\phi'_n\mathcal{B} + \sum_{k=0}^{n} \hat{\chi}_k(\mathcal{B})\phi_{n-k}, \quad n = 1, 2, \ldots. \end{aligned} \tag{A.6}$$

Example 3. If we set $\phi_n = -\hat{\chi}_n(\phi)$, $n = 0, 1, \ldots$, and hence

$$\mathcal{E}(\lambda) = I + \lambda\phi_{-[\lambda]}, \tag{A.7}$$

then the subsystem of (A.2) for $m = 0$ is

$$\hat{\chi}_{n+1}(\phi) + \hat{\chi}_n(\phi_x + \phi^2) - \hat{\chi}_n(\phi)\phi = 0, \quad n = 0, 1, \ldots, \tag{A.8}$$

which in functional form after a Miwa shift becomes representation (4.16) of the "opposite" Burgers hierarchy. The remaining equations resulting from (A.2) are

$$\hat{\chi}_m\hat{\chi}_{n+1}(\phi) - \hat{\chi}_n\hat{\chi}_{m+1}(\phi) = \sum_{k=1}^{m} \hat{\chi}_{m-k}\hat{\chi}_n(\phi)\hat{\chi}_k(\phi) - \sum_{k=1}^{n} \hat{\chi}_{n-k}\hat{\chi}_m(\phi)\hat{\chi}_k(\phi),$$

where $m, n = 1, 2, \ldots$. By the Hasse–Schmidt derivation property of the $\hat{\chi}_n$, this is form (4.7) of the pKP hierarchy. But we already know that the pKP hierarchy is satisfied as a consequence of the Burgers hierarchy. Equations (A.4) become

$$\phi = -f_x f^{-1}, \qquad \hat{\chi}_n(\phi) = \hat{\chi}_{n+1}(f) f^{-1}, \quad n = 1, 2, \ldots. \tag{A.9}$$

This leads to the linear functional equation

$$f^{-1}_{[\lambda]} = f^{-1} + \lambda (f^{-1})_x, \tag{A.10}$$

and hence $\chi_n(f^{-1}) = 0$ for $n = 2, 3, \ldots$, which is equivalent to the version of a linear heat hierarchy

$$\partial_{t_n}(f^{-1}) = (-1)^{n+1} \partial_x^n (f^{-1}), \quad n = 2, 3, \ldots. \tag{A.11}$$

As a consequence, if f^{-1} solves linear hierarchy (A.11), then $\phi = -f_x f^{-1}$ solves Burgers hierarchy (4.16) and also the pKP hierarchy.

Equations (A.6) become

$$\phi' = \mathcal{B} \phi \mathcal{B}^{-1} - \mathcal{B}_x \mathcal{B}^{-1}, \qquad (I + \lambda \phi') \mathcal{B}_{[\lambda]} = \mathcal{B}(I + \lambda \phi). \tag{A.12}$$

Using the first equation in the second to eliminate ϕ' yields an equation linear in $\mathcal{B}^{-1}$,

$$(\lambda^{-1} + \phi)\big(\mathcal{B}^{-1}_{[\lambda]} - \mathcal{B}^{-1}\big) = (\mathcal{B}^{-1})_x. \tag{A.13}$$

Comparing this with Burgers hierarchy system (4.16) shows that $\mathcal{B}^{-1} = \phi$ is a solution. More generally, $\mathcal{B}^{-1} = \alpha + \phi\beta$ with any constant α and β solves this equation.

Example 4. Setting $\phi_n = 0$ for $n > 0$ and $\phi := \phi_0$ reduces hierarchy (A.2) to the Burgers hierarchy in Sec. 3, and the second equation in (A.4) requires that f solve the linear heat hierarchy. If we relax the constraint to $\phi_n = 0$ for $n > 1$, thus leaving ϕ_0 and ϕ_1 as dependent variables, then (A.2) results in

$$\begin{aligned} &\big(\hat{\chi}_{n+1}(\phi_0) - \hat{\chi}_n(\phi_0)\phi_0 - \hat{\chi}_{n-1}(\phi_0)\phi_1\big)\delta_{m,0} + \\ &\quad + \big(\hat{\chi}_{n+1}(\phi_1) - \hat{\chi}_n(\phi_1)\phi_0 - \hat{\chi}_{n-1}(\phi_1)\phi_1\big)\delta_{m,1} = \\ &\qquad = \big(\hat{\chi}_{m+1}(\phi_0) - \hat{\chi}_m(\phi_0)\phi_0 - \hat{\chi}_{m-1}(\phi_0)\phi_1\big)\delta_{n,0} + \\ &\qquad\quad + \big(\hat{\chi}_{m+1}(\phi_1) - \hat{\chi}_m(\phi_1)\phi_0 - \hat{\chi}_{m-1}(\phi_1)\phi_1\big)\delta_{n,1}. \end{aligned} \tag{A.14}$$

It suffices to consider $m < n$. For $m = 0$ and $n = 1$, this yields

$$\phi_{0,y} - \phi_{0,xx} - 2\phi_{0,x}\phi_0 = 2\phi_{1,x} + 2[\phi_1, \phi_0]. \tag{A.15}$$

The equations of system (A.14) for $m = 0$ and $n > 1$ are

$$\hat{\chi}_{n+1}(\phi_0) - \hat{\chi}_n(\phi_0)\phi_0 - \hat{\chi}_{n-1}(\phi_0)\phi_1 = 0, \quad n = 2, 3, \ldots, \tag{A.16}$$

and for $m = 1$ and $n > 1$ are

$$\hat{\chi}_{n+1}(\phi_1) - \hat{\chi}_n(\phi_1)\phi_0 - \hat{\chi}_{n-1}(\phi_1)\phi_1 = 0, \quad n = 2, 3, \ldots. \tag{A.17}$$

In the case under consideration, Eqs. (A.4) become

$$\phi_0 = f_x f^{-1}, \qquad \phi_1 = -\hat{\chi}_2(f) f^{-1} = \frac{1}{2}(f_y - f_{xx}) f^{-1}, \tag{A.18}$$

and

$$\hat{\chi}_n(f) = 0, \quad n = 3, 4, \ldots, \tag{A.19}$$

which is *not* equivalent to the heat hierarchy, because $\hat{\chi}_2(f) = 0$ is missing.

REFERENCES

1. A. V. Mikhailov, A. B. Shabat, and V. V. Sokolov, "The symmetry approach to classification of integrable equations," in: *What Is Integrability?* (V. E. Zakharov, ed.), Springer, Berlin (1991), p. 115–184.
2. D. V. Choodnovsky and G. V. Choodnovsky, *Nuovo Cimento B*, **40**, 339–352 (1977).
3. D. Levi, O. Ragnisco, and M. Bruschi, *Nuovo Cimento B*, **74**, 33–51 (1983).
4. M. Bruschi and O. Ragnisco, *J. Math. Phys.*, **26**, 943–945 (1985).
5. A. Pickering, *J. Math. Phys.*, **35**, 821–833 (1994).
6. A. S. Fokas and Q. M. Liu, *Phys. Rev. Lett.*, **72**, 3293–3296 (1994).
7. H. Tasso, *J. Phys. A*, **29**, 7779–7784 (1996).
8. B. A. Kupershmidt, *KP or mKP*: *Noncommutative Mathematics of Lagrangian, Hamiltonian, and Integrable Systems* (Math. Surveys Monogr., Vol. 78), Amer. Math. Soc., Providence, R. I. (2000).
9. B. A. Kupershmidt, *J. Nonlinear Math. Phys.*, **12**, 539–549 (2005).
10. A. Dimakis and F. Müller-Hoissen, "Nonassociativity and integrable hierarchies," arXiv:nlin/0601001v2 [nlin.SI] (2006).
11. F. Guil, M. Mañas, and G. Álvarez, *Phys. Lett. A*, **190**, 49–52 (1994).
12. A. Dimakis and F. Müller-Hoissen, *J. Phys. A*, **39**, 9169–9186 (2006).
13. A. Dimakis and F. Müller-Hoissen, *J. Phys. A*, **39**, 14015–14033 (2006).
14. M. Gekhtman and A. Kasman, *J. Geom. Phys.*, **56**, 282–309 (2006).
15. M. Sato and Y. Sato, "Soliton equations as dynamical systems on infinite dimensional Grassmann manifold," in: *Nonlinear Partial Differential Equations in Applied Science* (Math. Stud., Vol. 81, H. Fujita, P. D. Lax, and G. Strang, eds.), North-Holland, Amsterdam (1983), p. 259–271.
16. E. Date, M. Jimbo, and T. Miwa, *J. Phys. Soc. Japan*, **51**, 4116–4124 (1982).
17. V. E. Adler, A. I. Bobenko, and Yu. B. Suris, *Comm. Math. Phys.*, **233**, 513–543 (2003).
18. A. I. Bobenko and Yu. B. Suris, *Lett. Math. Phys.*, **61**, 241–254 (2002).
19. A. Dimakis, F. Müller-Hoissen, and T. Striker, *J. Phys. A*, **26**, 1927–1949 (1993).
20. A. Dimakis, F. Müller-Hoissen, and T. Striker, *Phys. Lett. B*, **300**, 141–144 (1993).
21. L.-L. Chau, J. C. Shaw, and H. C. Yen, *Comm. Math. Phys.*, **149**, 263–278 (1992).
22. E. Hopf, *Comm. Pure Appl. Math.*, **3**, 201–230 (1950).
23. J. D. Cole, *Quart. Appl. Math.*, **9**, 225–236 (1951).
24. J. Weiss, M. Tabor, and G. Carnevale, *J. Math. Phys.*, **24**, 522–526 (1983).
25. K. T. Joseph and A. S. Vasudeva Murthy, *NoDEA Nonlinear Differential Equations Appl.*, **8**, 173–193 (2001).
26. D. J. Arrigo and F. Hickling, *J. Phys. A*, **35**, L389–L399 (2002).
27. M. Hamanaka and K. Toda, *J. Phys. A*, **36**, 11981–11998 (2003).
28. L. Martina and O. K. Pashaev, "Burgers' equation in noncommutative space–time," arXiv:hep-th/0302055v1 (2003).
29. A. S. Fokas, "Invariants, Lie–Bäcklund operators, and Bäcklund transformations," Doctoral dissertation, California Inst. Technology, Pasadena, Calif. (1979).
30. V. B. Matveev and M. A. Salle, *Darboux Transformations and Solitons*, Springer, Berlin (1991).
31. L. V. Bogdanov and B. G. Konopelchenko, *J. Math. Phys.*, **39**, 4701–4728 (1998).

32. L. V. Bogdanov, *Analytic-Bilinear Approach to Integrable Hierarchies* (Math. Appl., Vol. 493), Kluwer, Dordrecht (1999).
33. J. Dorfmeister, E. Neher, and J. Szmigielski, *Quart. J. Math.*, **40**, 161–195 (1989).
34. V. B. Matveev, *Lett. Math. Phys.*, **3**, 213–216 (1979).
35. W. Oevel, *Phys. A*, **195**, 533–576 (1993).
36. Q. P. Liu and M. Mañas, *J. Nonlinear Sci.*, **9**, 213–232 (1999).
37. W. T. Reid, *Riccati Differential Equations* (Math. Sci. Eng., Vol. 86), Acad. Press, New York (1972).
38. H. Abou-Kandil, G. Freiling, V. Ionescu, and G. Jank, *Matrix Riccati Equations in Control and Systems Theory*, Birkhäuser, Basel (2003).
39. S. V. Manakov, V. E. Zakharov, L. A. Bordag, A. R. Its, and V. B. Matveev, *Phys. Lett. A*, **63**, 205–206 (1977).
40. G. Biondini and S. Chakravarty, *J. Math. Phys.*, **47**, 033514 (2006).
41. A. Dimakis and F. Müller-Hoissen, *Czech. J. Phys.*, **56**, 1123–1130 (2006); arXiv:nlin/ 0608017v1 [nlin.PS] (2006).